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THE BLOW-UP SURFACE FOR NONLINEAR WAVE EQUATIONS WITH SMALL SPATIAL VELOCITY

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ABSTRACT. Consider the Cauchy problem for $u_{tt} - \varepsilon^2 \Delta u = f(u)$ in space dimension ≤ 3 where $f(u)$ is superlinear and nonnegative. The solution blows up on a surface $t = \phi_\varepsilon(x)$. Denote by $t = \phi(x)$ the blow-up surface corresponding to $v'' = f(v)$. It is proved that $|\phi_\varepsilon(x) - \phi(x)| \leq C\varepsilon^2$, $|\nabla(\phi_\varepsilon(x) - \phi(x))| \leq C\varepsilon^2$ in a neighborhood of any point x_0 where $\phi(x_0) < \infty$.

1. The main results. Let

$$\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}, \qquad \Box_\varepsilon = \frac{\partial^2}{\partial t^2} - \varepsilon^2 \Delta \qquad (\varepsilon > 0)$$

and consider the Cauchy problem

$$\Box_\varepsilon u_\varepsilon = f(u_\varepsilon) \quad \text{in } \mathbf{R}^N \times (0, \infty), \tag{1.1}$$

$$u_\varepsilon(x, 0) = g(x), \qquad x \in \mathbf{R}^N, \tag{1.2}$$

$$\frac{\partial}{\partial t} u_\varepsilon(x, 0) = h(x), \qquad x \in \mathbf{R}^N. \tag{1.3}$$

Here $f(u)$ is a superlinear function such as $(u^+)^p$; more generally we shall assume that $f \geq 0$, $f \in C^4(\mathbf{R})$; there exists a $u_0 \geq 0$ such that

$$\begin{aligned} &f(u) > 0, \quad f'(u) \geq 0, \quad f''(u) \geq 0 \qquad \text{if } u \geq u_0; \\ &(f(u)/u^p) \to 1 \quad \text{if } u \to \infty, \ p > 1; \\ &\limsup_{u \to \infty} (f'(u)/u^{p-1}) < p + (p-1)/2; \\ &\liminf_{u \to \infty} (f'(u)/u^{p-1}) > 0; \\ &|f^{(j)}(u)| \leq C u^{p-j} \quad \text{if } u \geq u_0, \ 2 \leq j \leq 4. \end{aligned} \tag{1.4}$$

We also assume that

$$N \leq 3 \tag{1.5}$$

and

$$\begin{aligned} g \in C^5(\mathbf{R}^N), \ h \in C^4(\mathbf{R}^N) \quad &\text{if } N = 2, 3; \\ g, h \in C^4(\mathbf{R}^1) \quad &\text{if } N = 1. \end{aligned} \tag{1.6}$$

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Sufficient conditions for nonexistence of global solutions of (1.1)–(1.3) are given in [**3**–**7**]. In this paper we are interested in the behavior of (blowing-up) solutions u_ε of (1.1)–(1.3) as $\varepsilon \to 0$. This is naturally related to the behavior of the solutions of the ordinary differential equation

$$\frac{d^2u}{dt^2} = f(u) \quad \text{for } t > 0 \tag{1.7}$$

under the Cauchy conditions

$$u(x,0) = g(x), \qquad u_t(x,0) = h(x). \tag{1.8}$$

For each fixed x the solution of (1.7), (1.8) exists for $0 < t < \phi(x)$ where either $\phi(x) = \infty$ or $\phi(x) < \infty$; in the latter case it can be shown (see §2) that $u(x,t) \to \infty$ if $t \to \phi(x)$, and we say that $u(x,t)$ *blows up* at time $t = \phi(x)$. The surface $\{t = \phi(x)\}$ is called the *blow up surface* for u.

Caffarelli and Friedman [**1**] proved that if $N = 1$ then there exists a unique classical solution of (1.1)–(1.3) for all $0 < t < \phi_\varepsilon(x)$ where either $\phi_\varepsilon(x) \equiv \infty$ (no blow-up) or else $\phi_\varepsilon(x) < \infty$ for all $x \in \mathbf{R}^1$ and $\phi_\varepsilon \in C^1$, $|\phi'_\varepsilon(x)| < 1/\varepsilon$; further, $u_\varepsilon(x,t) \to \infty$ if $t \to \phi_\varepsilon(x)$. In [**2**] they extended these results to $N = 2, 3$ under some restrictions on the Cauchy data (in addition to (1.6)). We shall recall a slightly simplified version of their result in case $\varepsilon = 1$; this will be needed in the sequel.

Introduce the sets

$$\begin{aligned} &K^\varepsilon(x_0,t_0) = \{(x,t); |x - x_0| \le \varepsilon(t_0 - t),\ 0 \le t < t_0\},\\ &B_R(x_0) = \{|x - x_0| < R\}, \qquad B_R = B_R(0),\\ &K^\varepsilon_{R,T} = \bigcup_{x \in B_R} K^\varepsilon(x,T). \end{aligned}$$

We shall assume, in addition to (1.5), (1.6), the following conditions:

(1.9) the solution w of $w''(t) = f(w)$, $t > 0$ with $w(0) = w'(0) = \gamma$ blows up in finite time T, where $\gamma > u_0, T > 0$;

$$\begin{aligned} g(x) \ge 2\gamma,\ h(x) \ge 2\gamma \quad &\text{in } B_{R+T},\\ |\nabla g| + |\nabla^2 g| + |\nabla h| < \eta \quad &\text{in } B_{R+T},\ n > 0. \end{aligned} \tag{1.10}$$

THEOREM 1.1 [**2**]. *If η is sufficiently small, depending on R, γ, T, then there exists a classical solution $u_1(x,t)$ of (1.1)–(1.3) with $\varepsilon = 1$ in $K^1_{R,T} \cap \Omega$ where $\Omega = \{(x,t); x \in B_{R+T},\ 0 < t < \phi_1(x)\}$, and it satisfies*

(i) $0 < \phi_1(x) < T$,

(ii) $u_1(x,t) \to \infty$ *if* $t \to \phi_1(x) - 0$,

(iii) $\phi_1 \in C^1(B_{R+T})$ *and* $|\nabla \phi_1(x)| < 1$. *The solution is unique in $K^1_{R,T}$ and it belongs to $C^{3,1}$.*

The proof of existence of u_1 begins by constructing a sequence of finite valued solutions U_n where $U_0 = 0$ and

$$\begin{aligned} &\Box_1 U_{n+1} = f(U_n) \quad \text{in } \mathbf{R}^N \times (0,\infty),\\ &U_{n+1}(x,0) = g(x), \quad \frac{\partial}{\partial t} U_{n+1}(x,0) = h(x) \qquad (x \in \mathbf{R}^N). \end{aligned} \tag{1.11}$$

One shows that

$$U_n(x,t) \le U_{n+1}(x,t) \quad \text{in } K^1_{R,T} \tag{1.12}$$

and that $U_{n+1}(x,t) \to u_1(x,t)$ as $n \to \infty$, where u_1 satisfies the properties asserted in Theorem 1.1.

Consider the case

$$\phi(0) < \infty. \tag{1.13}$$

In §2 we shall prove

LEMMA 1.2. *If* (1.13) *holds then there exists an* $R' > 0$ *such that*

$$0 < \phi(x) < \infty \quad \textit{if } x \in B_{R'}, \tag{1.14}$$

$$\phi \in C^1(B_{R'}). \tag{1.15}$$

Actually ϕ belongs to $C^4(B_{R'})$, but this fact will not be needed.

In §3 we shall prove

LEMMA 1.3. *Fix any* T *such that* $\phi(0) < T < \infty$. *Then there exist* $R > 0$, $C > 0$ *and* $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon < \varepsilon_0$ *then there exists a unique solution* u_ε *of* (1.1)–(1.3) *in* $K^\varepsilon_{R,T} \cap \Omega^\varepsilon_{R,T}$ *where* $\Omega^\varepsilon_{R,T} = \{(x,t); x \in B_{R+T},\ 0 < t < \phi_\varepsilon(x)\}$ *and*

(i) $0 < \phi_\varepsilon(x) < T$,

(ii) $u_\varepsilon(x,t) \to \infty$ *if* $t \to \phi_\varepsilon(x) - 0$, $x \in B_R$,

(iii) $\phi_\varepsilon \in C^1(B_{R+T})$ *and* $|\nabla\phi_\varepsilon(x)| \le C$; *the solution belongs to* $C^{3,1}$.

We can now state the main results of this paper in case $\phi(0) < \infty$.

THEOREM 1.4. *If* (1.4)–(1.6) *hold and* $\phi(0) < \infty$ *then there exist positive contants* R, C *such that, for all* ε *sufficiently small,*

$$\sup_{B_R} |\phi_\varepsilon(x) - \phi(x)| \le C\varepsilon^2, \tag{1.16}$$

$$\sup_{B_R} |\nabla(\phi_\varepsilon(x) - \phi(x))| C\varepsilon^2. \tag{1.17}$$

Theorem 1.4 will be proved in §§4–6.

Observe that Lemma 1.2 implies that the set $D = \{x \in \mathbf{R}^N,\ \phi(x) < \infty\}$ if open, and Theorem 1.4 implies that the solution u_ε exists for $0 < t < \phi_\varepsilon(x)$ and x in any compact subset D_0 of D; further

$$|\phi_\varepsilon(x) - \phi(x)| \le C\varepsilon^2 \quad \forall x \in D_0,$$
$$|\nabla(\phi_\varepsilon(x) - \phi(x))| \le C\varepsilon^2 \quad \forall x \in D_0.$$

In §7 we shall consider the case $\phi(0) = \infty$ and derive growth rates for $\phi_\varepsilon(0)$ as $\varepsilon \to 0$.

2. The equation $u'' = f(u)$. Throughout §§2–6 we assume that (1.13) holds. Set

$$F(u) = \int_0^u f(s)\,ds.$$

From (1.7), (1.8) we obtain

$$u_t^2 - h^2(x) = 2[F(u) - F(g(x))] \quad \text{if } t < \phi(x), \tag{2.1}$$

and then also

$$\phi(x) = \int_{g(x)}^{\infty} \frac{du}{[2F(u) - 2F(g(x)) + h^2(x)]^{1/2}} \tag{2.2}$$

provided, say, $g(x) > u_0$ (so that the denominator in the integrand is well defined).

Set

$$T(\gamma, \delta) = \int_{\gamma}^{\infty} \frac{du}{[2F(u) - 2F(\gamma) + \delta^2]^{1/2}} \tag{2.3}$$

for $\gamma > u_0$, $\delta > 0$. Then $T(\gamma, \delta)$ is the blow-up time of (1.7) subject to $u(0) = \gamma$, $u'(0) = \delta$. Using (1.4) we can easily show that $u'(t) > 0$, $u''(t) > 0$ if $t > 0$ and then, from the differential equation for $\partial u/\partial\gamma$, $\partial u/\partial\gamma$ remains positive for all $0 < t < T(\gamma, \delta)$. It follows that $\partial T/\partial\gamma \leq 0$. Since

$$\frac{\partial T}{\partial \gamma} = -\frac{1}{\delta} + f(\gamma) \int_{\gamma}^{\infty} \frac{du}{[2F(u) - 2F(\gamma) + \delta^2]^{3/2}},$$

we deduce that

$$|\partial T/\partial\gamma| \leq 1/\delta. \tag{2.4}$$

Next

$$\frac{\partial T}{\partial \delta} = -\delta \int_{\gamma}^{\infty} \frac{du}{[2F(u) - 2F(\gamma) + \delta^2]^{3/2}}$$

so that

$$|\partial T/\partial\delta| \leq T/\delta. \tag{2.5}$$

The assumption $\phi(0) < \infty$ implies that

$$u(0,t) \to \infty \quad \text{if } t \to \phi(0). \tag{2.6}$$

Indeed, if $u(0, t_n)$ remains bounded for a sequence $t_n \to \phi(0)$, then, by (2.1), also $u_t(0, t_n)$ remains bounded. But then the solution $u(0,t)$ of $u_{tt} = f(u)$ can be extended to $t_n < t < t_n + \delta$ with δ positive and independent of n, which is a contradiction if n is large enough.

From (2.6) and (2.1) we get

$$u_t(0,t) \to \infty \quad \text{if } t \to \phi(0). \tag{2.7}$$

Consequently, for any $\gamma > u_0$ there exists a $t_0 \in (0, \phi(0))$ such that $u(0, t_0) > \gamma$, $u_t(0, t_0) > \gamma$ and, by continuity

$$u(x, t_0) > \gamma, \quad u_t(x, t_0) > \gamma \qquad \text{if } x \in B_{R_0} \tag{2.8}$$

for some $R_0 > 0$. Using (1.4) we easily deduce that $u(x,t)$ blows up in finite time $\phi(x)$ for any $x \in B_{R_0}$. Further, analogously to (2.2), we have

$$\phi(x) = t_0 + T(u(x, t_0),\ u_t(x, t_0)), \qquad x \in B_{R_0}. \tag{2.9}$$

Since $u(x, t_0)$ and $u_t(x, t_0)$ vary smoothly with x and since (2.4), (2.5) hold, we conclude:

LEMMA 2.1. $\phi \in C^1(B_{R_0})$.

Set

$$\Omega_\rho = \{(x,t);\ x \in B_\rho,\ 0 \le t < \phi(x),\ \rho < R_0\}.$$

LEMMA 2.2. *For any $0 < R < R_0$ there exist positive constants C and c such that*

$$|D^\alpha u(x,t)| \le C(\phi(x)-t)^{-(pq+|\alpha|-2)} \quad \text{in } \Omega_R \tag{2.10}$$

where $q = 2/(p-1)$, $0 \le |\alpha| \le 2$, and

$$c(\phi(x)-t)^{-(pq+j-2)} \le D_t^j u(x,t) \le C(\phi(x)-t)^{-(pq+j-2)} \quad \text{in } \Omega_R \tag{2.11}$$

for $0 \le j \le 3$.

PROOF. From (2.1), by integration,

$$\int_{u(x,t)}^{\infty} \frac{du}{[h^2(x) + 2F(u) - 2F(g(x))]^{1/2}} = \phi(x) - t.$$

Since $F(u) \sim u^{p+1}/(p+1)$ as $u \to \infty$, the estimate (2.11) for u readily follows. Next using (2.1) we can establish (2.11) for $D_t u$, and using (1.7) we can further establish (2.11) for $j = 2$ and then (from $u_{ttt} = f'(u)u_t$) for $j = 3$.

To prove (2.10) we introduce (cf. [**2**]) the functions

$$J_1 = C_1 u_t \pm D^\alpha u, \quad |\alpha| = 1,$$
$$J_2 = C_2 u_{tt} \pm D^\alpha u, \quad |\alpha| = 2,$$

with C_1, C_2 positive constants. For any $x_0 \in B_R$ we can choose $\delta > 0$ and $t_1 \in (0, \phi(x_0))$ such that $u(x,t_1) > u_0$ and $D_t^j u(x,t_1) > 1$ ifx $\in B_\delta(x_0)$ $(0 \le j \le 3)$. Hence, if C_1 is large enough then $J_1(x,t_1) > 0$ and $J_{1,t}(x,t_1) > 0$ for $x \in B_\delta(x_0)$. Since

$$d^2 J_1/dt^2 = f'(u) J_1,$$

we can easily deduce by a continuity argument that $J_1(x,t)$ remains positive for $t_1 < t < \phi(x)$, if $x \in B_\delta(x_0)$.

Next we choose C_2 such that $J_2(x,t_1) > 0$ and $J_{2,t}(x,t_1) > 0$ for $x \in B_\delta(x_0)$. We have

$$\frac{d^2 J_2}{dt^2} = f'(u) J_2 + f''(u)(C_2 u_t^2 \pm D^{\beta_1} u D^{\beta_2} u)$$

where $\beta_1 + \beta_2 = \alpha$. Since $J_1 > 0$, if C_2 is large enough then the coefficient of $f''(u)$ is positive. Hence, by a continuity argument, $J_2(x,t) > 0$ if $x \in B_\delta(x_0)$, $t_1 < t < \phi(x)$. Combining the positivity of J_1, J_2 with (2.11), the estimate (2.10) follows.

3. Proof of Lemma 1.3. In the sequence we shall need an integral representation for solutions of the inhomogeneous wave equation. The formula has a different form depending on the space dimension N. We shall consider only the case $N = 3$; the cases $N = 1, 2$ can be treated in a similar way.

For $N = 3$ we have

$$\begin{aligned} w(x,t) = {} & \frac{t}{4\pi}\int_{|\xi|=1} w_1(x+\varepsilon t\xi)\, d\omega_\xi + \frac{\partial}{\partial t}\frac{t}{4\pi}\int_{|\xi|=1} w_0(x+\varepsilon t\xi)\, d\omega_\xi \\ & + \frac{1}{4\pi}\int_0^t (t-s)\, ds \int_{|\eta|=1} h(x+\varepsilon(t-s)\eta, s)\, d\omega_\eta \end{aligned} \tag{3.1}$$

where

$$w_0(y) = w(y,0), \qquad w_1(y) = w_t(y,0), \qquad h(y,t) = \Box_\varepsilon w(y,t).$$

For any $R_1 > 0$, $0 < \varepsilon < 1$ we can construct a solution u_ε of (1.1)–(1.3) in $K^\varepsilon_{R_1,\sigma_1}$ provided σ_1 is sufficiently small, independently of ε. In fact, define an operator S by

$$(Sw)(x,t) = G(x,t) + \frac{1}{4\pi}\int_0^t (t-s)\,ds \int_{|\eta|=1} f(w(x+\varepsilon(t-s)\eta, s))\,d\omega_\eta, \tag{3.2}$$

where

$$G(x,t) = \frac{t}{4\pi}\int_{|\xi|=1} h(x+\varepsilon t\xi)\,dw_\xi + \frac{\partial}{\partial t}\frac{t}{4\pi}\int_{|\xi|=1} g(x+\varepsilon t\xi)\,d\omega_\xi. \tag{3.3}$$

The domain of S is taken to be

$$X_{\sigma_1,M_1} = \{w \in C^0(K^\varepsilon_{R_1,\sigma_1}); \sup|w| \le M_1\}$$

where

$$M_1 = 1 + \sup_{K^\varepsilon_{R_1,T}} |G|.$$

Then, if σ_1 is sufficiently small, S maps X_{σ_1,M_1} into itself and it is a contraction. It follows that S has a fixed point, which is clearly a solution u_ε of (1.1)–(1.3). If $\tilde{u}_\varepsilon$ is another solution in K_{R_1,σ_2} then we easily deduce that

$$\|S\tilde{u}_\varepsilon - Su_\varepsilon\| < \tfrac{1}{2}\|\tilde{u}_\varepsilon - u_\varepsilon\|$$

where the norm is the supremum norm in $K_{R_1,\sigma}$ with σ small enough, depending on $\|\tilde{u}_\varepsilon\|$. It follows that $\tilde{u}_\varepsilon = u_\varepsilon$ if $0 < t < \sigma$, and by a step-by-step argument, also if $0 < t < \min(\sigma_1, \sigma_2)$.

Since $\Box_\varepsilon(DU_\varepsilon) = f'(u_\varepsilon)(Du_\varepsilon)$, we can apply (3.1) to Du_ε and, by estimating successively the right-hand side, we find that

$$|Du_\varepsilon| \le M_1' \quad \text{in } K^\varepsilon_{R_1,\sigma_1}$$

provided σ_1 is small enough; σ_1 and M_1' are independent of ε. Similarly

$$|D^\alpha u_\varepsilon| \le M_1' \quad \text{in } K^\varepsilon_{R_1,\sigma_1}, \ |\alpha| \le 3, \tag{3.4}$$

with another constant M_1'.

From (3.4) it follows that any sequence $\varepsilon \to 0$ has a subsequence such that

(3.5) $D^\alpha u_\varepsilon \to D^\alpha v$ in $(L^\infty(K^\varepsilon_{R_1,\sigma_1}))^*$ weakly; $0 \le |\alpha| \le 3$, and, therefore

(3.6) $D^\alpha u_\varepsilon \to D^\alpha v$ uniformly in compact subsets of $\bigcap_{\varepsilon'} K^{\varepsilon'}_{R_1,\sigma_1}$; $0 \le |\alpha| \le 2$.

Hence

$$v(x,t) = u(x,t) \tag{3.7}$$

where u is the solution of (1.7), (1.8) and (3.5), (3.6) hold for all $\varepsilon \to 0$.

We wish to extend the solution u_ε beyond $t = \sigma_1$. To do this we repeat the previous proof, considering S in the space

$$X_{\sigma_2,M_2} = \{w \in C^0(K^\varepsilon_{R_2,\sigma_2}), w \equiv u_\varepsilon \text{ in } K^\varepsilon_{R_2,\sigma_2} \cap \{t < \sigma_1\}, \sup|w| \le M_1\}$$

for any $R_2 < R_1$, $M_2 = M_1 + 1$ and some $\sigma_2 > \sigma_1$. Then S is a contraction if $\sigma_2 - \sigma_1$ is sufficiently small, depending on M_2. As before we can establish (3.5)–(3.7) in $K^{\varepsilon}_{R_2,\sigma_2}$.

We can carry out the above procedure with $\sigma_3, R_3, M_3, \sigma_4, R_4, M_4$, etc.; however, the numbers $\sigma_{j+1} - \sigma_j$ are decreasing since the M_j are increasing. Let

$$\tilde{\sigma} < \inf_{B_{R_1}} \phi(x), \qquad 0 < \tilde{R} < R_1,\ R_1 \text{ small.} \tag{3.8}$$

We claim that the above procedure yields, in a finite number j_0 of steps (j_0 independent of ε) a solution u_ε in $K_{\tilde{R},\tilde{\sigma}}$. Indeed, from (3.8) we have that

$$|D^\alpha u(x,t)| \leq C \quad \text{in } B_{R_1} \times [0, \tilde{\sigma}]\ (|\alpha| \leq 3).$$

Hence, in view of (3.5)–(3.7) we may repeat the previous construction of u_ε but with the following modifications: at each step j we must take $\varepsilon \leq \varepsilon_j$ so that for $|\alpha| \leq 2$ we have $|D^\alpha u_\varepsilon| < C + 1$ in K_{R_j,σ_j}. Hence we obtain the bound $C + 2$ instead of M_j for $|D^\alpha u_\varepsilon|$ in $K_{R_{j+1},\sigma_{j+1}}$ ($|\alpha| \leq 2$). Next, by Gronwall's inequality we can derive a bound M_{j+1}, independent of ε, for $|D^\alpha u_\varepsilon|$ in $K_{R_{j+1},\sigma_{j+1}}$ where $|\alpha| = 3$. We have $\varepsilon_1 > \varepsilon_2 > \cdots$. However, the differences $\sigma_{j+1} - \sigma_j$ remain uniformly positive independently of the choice of the R_j; say $\sigma_{j+1} - \sigma_j \geq \delta > 0$ (ε_j depends on R_j). Choosing $j_0 = [\tilde{\sigma}/\delta] + 1$, and $R_j = R_1 - (R_1 - \tilde{R})/j_0$, we obtain the desired solution u_ε in $K^{\varepsilon}_{\tilde{R},\tilde{\sigma}}$. Further,

(3.9) $D^\alpha u_\varepsilon \to D^\alpha u$ uniformly in compact subsets of $B_{\tilde{R}} \times [0, \tilde{\sigma}]$, $0 \leq |\alpha| \leq 2$.

Choosing R_1 sufficiently small, we can take $\tilde{\sigma}$ sufficiently close to $\phi(0)$. Hence in view of (2.8), (3.9) we have

$$u_\varepsilon(x, t_0) > 2\gamma,\ u_{\varepsilon,t}(x, t_0) > 2\gamma \quad \text{if } x \in B_{2R}, \tag{3.10}$$

provided $2R < \tilde{R}$, where $\gamma > u_0$ and t_0 is some point in $(0, \tilde{\sigma})$.

We now introduce the scaled functions

$$U_\varepsilon(x,t) = u_\varepsilon(\varepsilon x, t) \quad \text{for } t \geq t_0.$$

Then $\Box_1 U_\varepsilon = f(U_\varepsilon)$. Setting $g_\varepsilon(x) = U_\varepsilon(x, t_0)$, $h_\varepsilon(x) = U_{\varepsilon,t}(x, t_0)$, we have

$$|\nabla g_\varepsilon| + |\nabla h_\varepsilon| \leq C\varepsilon, \qquad |\nabla^2 g_\varepsilon| \leq C\varepsilon^2. \tag{3.11}$$

Using (3.10), (3.11) we can now apply the proof of Theorem 1.4 (as given in [**2**]) in order to establish the existence of a unique solution U_ε in $K^1_{(R/\varepsilon),T} \cap \{t \geq t_0\} \cap \{t < \tilde{\phi}_\varepsilon(x)\}$ for all ε small enough, and the estimate

$$U_{\varepsilon,t} \geq (c_0/\varepsilon)|\nabla_x U_\varepsilon|, \qquad c_0 > 0: \tag{3.12}$$

the function $U_\varepsilon(x/\varepsilon, t)$ is then the unique extension of the solution u_ε to $\{t < \phi_\varepsilon(x)\}$, where $\phi_\varepsilon(x) = \tilde{\phi}(\varepsilon x)$. It will be shown below that

$$\phi_\varepsilon(x) < T. \tag{3.13}$$

Then, the assertion (ii) of Lemma 1.3 follows (from the proof of Theorem 1.4 for U_ε), and $\phi_\varepsilon \in C^1$. Further, from (3.12) we deduce that $|\nabla \phi_\varepsilon| \leq 1/c_0$ and thus Lemma 1.3 follows.

To prove (3.13) let $W_\delta(t)$ (δ positive and small) be the solution of

$$\begin{aligned} W_\delta'' &= f(W_\delta) \quad \text{if } t > t_0, \\ W_\delta(t_0) &= u(0,t_0) - 2\delta, \\ W_\delta'(t_0) &= u_t(0,t_0) - 2\delta. \end{aligned} \tag{3.14}$$

We shall compare W_δ with U_ε in $K_{(R/\varepsilon),T}$. By (3.9), if R is small enough, depending on δ, then

$$\begin{aligned} W_\delta(t_0) &\le U_\varepsilon(x,t_0) - \delta, \\ W_\delta'(t_0) &\le U_{\varepsilon,t}(x,t_0) - \delta \quad \text{in } B_{(R/\varepsilon)+T} \end{aligned} \tag{3.15}$$

provided ε is sufficiently small. Also,

$$|\nabla U_\varepsilon(x,t_0)| \le C\varepsilon < \delta \quad \text{if } \varepsilon < \delta/C. \tag{3.16}$$

Hence, by a comparison argument based on the representation (3.1), (3.2) (cf. the proof of Lemma 2.3 in [**2**]), it follows that

$$W_\delta(t) \le U_\varepsilon(x,t) \quad \text{in } K_{(R/\varepsilon),T},$$

and thus $\phi_\varepsilon(x) < T_\delta$ where T_δ is the blow-up time for W_δ. By the results of §2 (cf. (2.4), (2.5)), $|T_\delta - \phi(x)| < C_1\delta$. Consequently

$$\phi_\varepsilon(x) \le \phi(x) + C_1\delta, \tag{3.17}$$

and (3.13) follows.

REMARK 3.1. If we replace $-\delta$ by $+\delta$ in (3.14) then the previous argument yields the estimate

$$\phi_\varepsilon(x) \ge \phi(x) - C_1\delta. \tag{3.18}$$

4. Estimating $u_\varepsilon - u$.

LEMMA 4.1. *If R is sufficiently small then for any compact subset D_0 of*

$$\Omega_R \equiv \{(x,t);\ x \in B_R,\ 0 \le t < \phi(x)\}$$

there exists a constant C such that

$$|u_\varepsilon - u| \le C\varepsilon^2 \quad \text{in } D_0, \tag{4.1}$$

$$|u_{\varepsilon,t} - u_t| \le C\varepsilon^2 \quad \text{in } D_0 \tag{4.2}$$

if ε is sufficiently small.

PROOF. By the estimates of §3 we deduce that if ρ is sufficiently small then

$$|D^\alpha u_\varepsilon| \le C \quad \text{for } 0 \le |\alpha| \le 3 \tag{4.3}$$

provided $(x,t) \in D_0$ and $|x| < \rho$. Similarly, for any $x_0 \in B_R$ the estimate (4.3) holds on $\{(x,t) \in D_0,\ x \in B_\rho(x_0)\}$, where C and ρ can be taken independently of x_0. It follows that (4.3) holds in D_0.

We can then write

$$\frac{\partial^2}{\partial t^2} u_\varepsilon = f(u_\varepsilon) + h_\varepsilon, \qquad |h_\varepsilon| = |\varepsilon^2 \Delta u_\varepsilon| \le C\varepsilon^2. \tag{4.4}$$

Representing u in the form

$$u(x,t) = g(x) + th(x) + \int_0^t (t-\tau) f(u(x,\tau))\, d\tau \tag{4.5}$$

and, similarly,

$$\begin{aligned} u_\varepsilon(x,t) &= g(x) + th(x) + \int_0^t (t-\tau) f(u_\varepsilon(x,\tau))\, d\tau \\ &\quad + \int_0^t (t-\tau) h_\varepsilon(x,\tau)\, d\tau \end{aligned} \tag{4.6}$$

and taking the difference, we get

$$|u_\varepsilon(x,t) - u(x,t)| \le C \int_0^t |u_\varepsilon(x,\tau) - u(x,\tau)| + C\varepsilon^2$$

provided D_0 is taken to be a subgraph in the t-direction, and (4.1) follows.

Similarly

$$\frac{\partial^2}{\partial t^2} u_{\varepsilon,t} = f'(u_\varepsilon) u_{\varepsilon,t} + h_1, \qquad |h_1| = |\varepsilon^2 \Delta u_{\varepsilon,t}| \le C\varepsilon^2$$

$$\frac{\partial^2}{\partial t^2} u_t = f'(u) u_t,$$

and

$$u_{\varepsilon,t}(x,0) = h(x) = u_t(x,0),$$
$$u_{\varepsilon,tt}(x,0) = \varepsilon^2 \Delta g + f(g) = u_{tt}(x,0) + \varepsilon^2 \Delta g$$

and the previous argument coupled with the estimate (4.1) yields the assertion (4.2).

5. Estimating $\phi_\varepsilon - \phi$ and estimating $D^\alpha(u_\varepsilon - u)$ near $\{t = \phi\}$.

LEMMA 5.1. *If R is sufficiently small then there exists a constant C such that*

$$\sup_{B_R} |\phi_\varepsilon(x) - \phi(x)| \le C\varepsilon \tag{5.1}$$

for all ε sufficiently small.

PROOF. We re-examine the proof of (3.17), (3.18). It is easy to see that (3.15) holds with $\delta = A\varepsilon$ provided A is a sufficiently large positive number. Recalling also (3.16), we deduce as before that (3.17) holds if $\delta = C\varepsilon$ and ε is sufficiently small. The proof of (3.18) with $\delta = C\varepsilon$ is similar.

In the sequel we shall need some estimates on $D^\alpha u_\varepsilon$ and $D^\alpha(u_\varepsilon - u)$ near the blow-up surface. We begin with

LEMMA 5.2. *There exist $t_1 \in (0, \phi(0))$ and $R > 0$ such that, for all ε sufficiently small,*

$$c(\phi_\varepsilon(x) - t)^{(pq+j-2)} \le D_t^j u_\varepsilon(x,t) \le C(\phi_\varepsilon(x) - t)^{-(pq+j-2)} \qquad (0 \le j \le 2), \tag{5.2}$$

$$|D^\alpha u_\varepsilon(x,t)| \le C(\phi_\varepsilon(x) - t)^{-(pq+|\alpha|-2)} \qquad (0 \le |\alpha| \le 2) \tag{5.3}$$

for $x \in B_R$, $t_1 < t < \phi_\varepsilon(x)$, where c, C are positive constants, and $q = 2/(p-1)$.

PROOF. To prove (5.2) we work with $U_\varepsilon(x,t) = u_\varepsilon(\varepsilon x, t)$ and establish for $D_t^j U_\varepsilon$ estimates (as in (5.2) with $\phi_\varepsilon(x)$ replaced by $\phi_\varepsilon(\varepsilon x)$) by the method of [**2**]. Since $D_t^j u_\varepsilon(x,t) = D_t^j U_\varepsilon(x/\varepsilon, t)$, the inequalities in (5.2) follow.

Consider next

$$J_1^\varepsilon = C_1 u_{\varepsilon,t} \pm D_x^\alpha u_\varepsilon \quad (|\alpha| = 1),$$
$$J_2^\varepsilon = C_2 u_{\varepsilon,tt} \pm D_x^\alpha u_\varepsilon \quad (|\alpha| = 2).$$

If C_1, C_2 are positive and sufficiently large, then

$$J_i^0(x,t_1) > 0,\ J_{i,t}^0(x,t_1) > 0 \quad \text{if } x \in B_R,$$

where J_i^0 is J_i^ε with u_ε replaced by u. Consequently, by (3.9),

$$J_i^\varepsilon(x,t_1) > 0,\ J_{i,t}^\varepsilon(x,t_1) > 0 \quad \text{if } x \in B_R \tag{5.4}$$

provided ε is sufficiently small. We can now proceed by a comparison argument as in [**2**] (cf. also the proof of Lemma 2.2) to show that if ε is small enough (so that also $|\nabla_x J_1^\varepsilon(\varepsilon x, t_1)| < \varepsilon J_{1,t}^\varepsilon(\varepsilon x, t_1)$, $\varepsilon x \in B_R$) then $J_1^\varepsilon(x,t) > 0$ in the set $(\bigcup_{x_0 \in B_r} K^\varepsilon(x_0, \phi_\varepsilon(x_0))) \cap \{t > t_1\}$. This yields the assertion (5.3) for $|\alpha| = 1$ (with a different R). Using this information we can next establish by comparison that $J_2^\varepsilon(x,t) > 0$ in the same domain as before provided C_2 is suitably chosen, and (5.3) thus follows for $|\alpha| = 2$.

By (5.1) we know that

$$|\phi_\varepsilon(x) - \phi(x)| \le C_0 \varepsilon \quad \text{if } x \in B_R.$$

We shall choose a constant M such that $M > 2C_0$. Then

$$\tilde{c} \le \frac{\phi_\varepsilon(x) - t}{\phi(x) - t} < \frac{1}{\tilde{c}} \quad \text{if } x \in B_R,\ 0 < t < \phi(x) - M\varepsilon \tag{5.5}$$

where $\tilde{c}$ is a positive constant independent of ε.

LEMMA 5.3. *The following estimates hold for* $t_1 < t < \phi(x) - M\varepsilon,\ x \in B_R$:

$$|D^\alpha(u_\varepsilon - u)(x,t)| \le C\varepsilon^2(\phi(x) - t)^{-(pq+|\alpha|-2)} \qquad (0 \le |\alpha| \le 2) \tag{5.6}$$

where C is a positive constant independent of ε.

PROOF. We proceed as in Lemma 4.1 but use the estimates of Lemma 5.2. From the integral representation of u_ε and u in (4.6) and (4.5) we obtain, by taking the difference,

$$|u_\varepsilon(x,t) - u(x,t)| \le C\varepsilon^2 \int_{t_1}^t \int_{t_1}^{\tilde{t}} |\Delta u_\varepsilon| + \int_{t_1}^t \int_{t_1}^{\tilde{t}} f'(\tilde{u})|u_\varepsilon - u| + C\varepsilon^2$$

where, in each integral, both integrations are in t, and $\tilde{u}$ lies between u and u_ε. Setting $\lambda = \phi(x) - t$ (x fixed) and using Lemma 5.2, we obtain

$$\begin{aligned}
|u_\varepsilon| &\le C\varepsilon^2 \int_{t_*}^\lambda \int_{t_*}^{\tilde{\lambda}} \frac{1}{\tilde{\lambda}^{pq}} + C \int_{t_*}^\lambda \int_{t_*}^{\tilde{\lambda}} \frac{|u_\varepsilon - u|}{\tilde{\lambda}^{(pq-2)(p-1)}} + C\varepsilon^2 \\
&\le \frac{C\varepsilon^2}{\lambda^{pq-2}} + C \int_{t_*}^\lambda \int_{t_*}^{\tilde{\lambda}} \frac{|u_\varepsilon - u|}{\tilde{\lambda}^2} \\
&\le \frac{C\varepsilon^2}{\lambda^{pq-2}} + \frac{C}{\lambda} \int_{t_*}^\lambda |u_\varepsilon - u| \qquad (\phi(x) - t_1 = t_*).
\end{aligned}$$

Hence the function

$$z = \int_{t_*}^{\lambda} |u_\varepsilon - u|$$

satisfies

$$z' - Cz/\lambda \le C\varepsilon^2/\lambda^{pq-2}, \qquad z(t_*) = 0,$$

from which we deduce that $z \le C\varepsilon^2/\lambda^{pq-3}$. Hence

$$|u_\varepsilon - u| \le C\varepsilon^2/(\phi(x) - t)^{pq-2}. \tag{5.7}$$

This establishes (5.6) for $|\alpha| = 0$.

To consider the case $|\alpha| = 1$ we first take $D^\alpha = D_t$. By differentiating the integral representation of u_ε and u with respect to t and taking the difference, we get

$$|u_{\varepsilon,t} - u_t| \le c\varepsilon \int_{t_1}^{t} |\Delta u_\varepsilon| + \int_{t_1}^{t} f'(\tilde{u})|u_\varepsilon - u|.$$

Using (5.7) and Lemma 5.2, we easily estimate the right-hand side, thereby deriving (5.6).

To estimate $D_x^\alpha(u_\varepsilon - u)$ for $|\alpha| = 1$ we apply D_x^α to the integral representation of $u_\varepsilon - u$ and obtain

$$|D_x^\alpha(u_\varepsilon - u)| \le C\varepsilon^2 \iint |D_x^\alpha \Delta u_\varepsilon| + \iint |f'(u_\varepsilon)D_x^\alpha u_\varepsilon - f'(u)D_x^\alpha u|.$$

Estimating

$$|[f'(u_\varepsilon) - f'(u)]D_x^\alpha u_\varepsilon|$$

by (5.7) and Lemma 5.2 we find that

$$|D_x^\alpha(u_\varepsilon - u)| \le \frac{C\varepsilon^2}{\lambda^{pq-1}} + C\int_{t_*}^{\lambda}\int_{t_*}^{\tilde{\lambda}} \frac{|D_x^\alpha(u_\varepsilon - u)|}{\tilde{\lambda}^{(pq-2)(p-1)}}.$$

We can now proceed as before to establish (5.6) (with $|\alpha| = 1$). Finally, the proof of (5.6) for $|\alpha| = 2$ is similar; we argue separately in the cases D_t^2, $D_t D_x^\alpha$ ($|\alpha| = 1$) and D_x^α ($|\alpha| = 2$).

Using Lemmas 5.2 and 5.3 we shall now complete the proof of (1.16).

LEMMA 5.4. *If R is sufficiently small then there exists a constant C such that*

$$\sup_{B_R} |\phi_\varepsilon(x) - \phi(x)| \le C\varepsilon^2 \tag{5.8}$$

for all ε sufficiently small.

PROOF. We repeat the proof of Lemma 5.1 choosing, in the comparison argument (3.14),

$$t_0 = \phi(0) - 3M\varepsilon$$

and working in the cone K with base $B_{5M\varepsilon^2}(0)$ on $t = t_0$ and vertex $(0, \phi(0) + 2M\varepsilon)$. Set $d = 3M\varepsilon$. By Lemmas 5.2 and 5.3,

$$\begin{aligned} \gamma_\varepsilon &\equiv u(0, t_0) + C_0\varepsilon^2/d^{pq-1} \ge u_\varepsilon(x, t_0) \quad \text{if } x \in B_{5M\varepsilon^2}(0), \\ \delta_\varepsilon &\equiv u_t(0, t_0) + C_1\varepsilon^2/d^{pq} \ge u_{\varepsilon,t}(x, t_0) \quad \text{if } x \in B_{5M\varepsilon^2}(0), \end{aligned} \tag{5.9}$$

if C_0, C_1 are sufficiently large positive constants. Let $W_\varepsilon(t)$ be the solution of

$$W_\varepsilon'' = f(W) \quad \text{if } t > t_0,$$
$$W_\varepsilon(t_0) = \gamma_\varepsilon, \quad W_\varepsilon'(t_0) = \delta_\varepsilon.$$

Set $U_\varepsilon(x,t) = u_\varepsilon(\varepsilon x, t)$. From the integral representation (3.1) and a comparison argument (as in [**2**]) we see that if

$$\begin{aligned} W_\varepsilon'(t_0) + (t-t_0)W_\varepsilon(t_0) > U_{\varepsilon,t}(x,t_0) + (t-t_0)U_\varepsilon(x,t_0) \\ + (t-t_0)|\nabla_x U_\varepsilon(x,t_0)| \end{aligned} \tag{5.10}$$

then

$$W_\varepsilon(t) > u_\varepsilon(x,t) \quad \text{in } K$$

and consequently

$$\phi_\varepsilon(0) > T_\varepsilon \tag{5.11}$$

where T_ε is the blow-up time of $W_\varepsilon(t)$. Since $t - t_0 \le 5M\varepsilon$ and $|\nabla_x U(x,t)| = \varepsilon|\nabla u_\varepsilon(\varepsilon x, t)|$, (5.10) is a consequence of Lemma 5.2 provided we choose C_1 to be sufficiently large (independently of ε).

Setting $\gamma = u(0,t_0)$, $\delta = u_t(0,t_0)$ and using (2.3), we compute that

$$\begin{aligned} \phi(0) - T_\varepsilon &= \int_\gamma^\infty [2F(u) - 2F(\gamma) + \delta^2]^{-1/2}\,du \\ &\quad - \int_{\gamma_\varepsilon}^\infty [2F(u) - 2F(\gamma_\varepsilon) + \delta_\varepsilon^2]^{-1/2} \\ &\le \int_\gamma^{\gamma_\varepsilon} [2F(u) - 2F(\delta) + \delta^2]^{1/2}\,du \\ &\quad + \int_{\gamma_\varepsilon}^\infty \frac{C\varepsilon^2 d^{-(3p+1)/(p-1)}}{[2F(u) - 2F(\gamma) + \delta^2]^{3/2}} \equiv I_1 + I_2. \end{aligned}$$

Clearly

$$I_1 \le C(\gamma_\varepsilon - \gamma)/\delta \le C\varepsilon^2.$$

Next, substituting $u = \gamma v$ into I_2, we get

$$I_2 \le \frac{C\gamma}{\gamma^{(p+1)3/2}} \frac{\varepsilon^2}{d^{(3p+1)/(p-1)}} \le C\varepsilon^2.$$

We have thus proved that $\phi(0) - T_\varepsilon \le C\varepsilon^2$. Combining this with (5.10), it follows that

$$\phi(0) - \phi_\varepsilon(0) \le C\varepsilon^2.$$

Similarly one proves that $\phi(0) - \phi_\varepsilon(0) \ge -C\varepsilon^2$, and thus $|\phi(0) - \phi_\varepsilon(0)| \le C\varepsilon^2$. Since the above proof applies about each point x in some neighborhood of $x = 0$, (5.8) follows.

6. Estimating $\nabla(\phi_\varepsilon - \phi)$. Denote by $\phi_m(x)$ and $\psi_m(x)$ the solutions of

$$u(x, \phi_m(x)) = m, \qquad u_t(x, \psi_m(x)) = m;$$

in view of Lemma 2.2, $\phi_m(x)$ and $\psi_m(x)$ are well defined for $x \in B_R$, provided m is sufficiently large; R is as usual a small enough positive number. Denote by

$N(x)$, $N_m(x)$ and $M_m(x)$ the unit normals in the positive t-direction of the surfaces $\{t = \phi(x)\}$, $\{t = \phi_m(x)\}$ and $\{t = \psi_m(x)\}$ respectively. Thus, for instance,

$$N_m(x) = [1 + |\nabla\phi_m(x)|^2]^{-1/2}(-\nabla\phi_m(x), 1).$$

For any $\eta > 0$, denote by $S_\eta(x)$ the set of all unit vectors $\tau = \tau(x)$ with

$$\tau \cdot N(x) \geq \eta. \tag{6.2}$$

We claim

$$|\nabla(\phi_m(x) - \phi(x))| \leq Cm^{-(3p+1)/2} \quad \text{if } x \in B_R. \tag{6.3}$$

Indeed, analogously to (2.9) (t_0 is taken close to $\phi(0)$) $\phi_m(x)$ is given by

$$\phi_m(x) = t_0 + \int_{u(x,t_0)}^{m} \frac{du}{[2F(u) - 2F(u(x,t_0)) + (u_t(x,t_0))^2]^{1/2}}.$$

Therefore

$$\begin{aligned}\nabla\phi_m(x) = &-\frac{\nabla u(x,t_0)}{u_t(x,t_0)} \\ &+ \int_{u(x,t_0)}^{m} \frac{[f(u(x,t_0))\nabla u(x,t_0) - u_t(x,t_0)\nabla u_t(x,t_0)]\,du}{[2F(u) - 2F(u(x,t_0)) + (u_t(x,t_0))^2]^{3/2}}.\end{aligned}$$

Since $\phi(x) = \phi_\infty(x)$, we deduce that

$$\nabla(\phi(x) - \phi_m(x)) = \int_m^\infty [\cdots]\,du$$

with the same integrand as in the preceding integral. Hence

$$|\nabla(\phi_m(x) - \phi(x))| \leq C\int_m^\infty \frac{du}{u^{(p+1)3/2}},$$

and (6.3) follows.

We shall next prove that

$$|\nabla(\psi_m(x) - \phi(x))| \leq Cm^{-(3p+1)/(p+1)} \quad \text{if } x \in B_R. \tag{6.4}$$

Indeed, we have

$$u_t^2 - u_t^2(x,t_0) + 2F(u(x,t_0)) = 2F(u)$$

and $F(u)$ has an inverse $G = F^{-1}$, well defined and smooth, for all $u = u(x,t)$ with $t > t_0$ (by (2.6), (2.7)). We can then write

$$u = G\big(\tfrac{1}{2}u_t^2 - \tfrac{1}{2}u_t^2(x,t_0) + F(u(x,t_0))\big),$$

and (1.7) takes the form

$$u_{tt} = f\big(G\big(\tfrac{1}{2}u_t^2 - \tfrac{1}{2}u_t^2(x,t_0) + F(u(x,t_0))\big)\big).$$

By integration we then obtain

$$\psi_m(x) = t_0 + \int_{u_t(x,t_0)}^{m} \frac{dv}{f(G(\frac{1}{2}v^2 + a(x)))}$$

where

$$a(x) = -\tfrac{1}{2}u_t^2(x,t_0) + F(u(x,t_0)).$$

The same formula holds for $\phi(x)$ with $m = \infty$. Taking the gradient of the difference, we get

$$|\nabla(\psi(x) - \psi_m(x))| \leq C\int_m^\infty \frac{dv}{(\frac{1}{2}v^2 + a(x))^{1+p/(p+1)}}$$

and (6.4) follows.

LEMMA 6.1. *There exist a positive constant c such that, for any $\eta \in (0,1)$, if*

$$\phi(x) - t < c\eta^{(p-1)/(3p+1)} \tag{6.5}$$

then

$$\tau \cdot N_{u(x,t)}(x) > \eta/3, \tag{6.6}$$

$$\tau \cdot M_{u_t(x,t)}(x) > \eta/3 \tag{6.7}$$

for any $x \in B_R$, $\tau \in S_{2\eta/3}(x)$.

PROOF. From (6.3), (6.4) and Lemma 2.2,

$$|\nabla(\phi_m - \phi)| \le C/u^{(3p+1)/2} \le C(\phi - t)^{(3p+1)/(p-1)}, \qquad m = u(x,t), \tag{6.8}$$

$$|\nabla(\psi_m - \phi)| \le C/u_t^{(3p+1)/(p+1)} \le C(\phi - t)^{(3p+1)/(p-1)}, \qquad m = u_t(x,t). \tag{6.9}$$

From (6.1) we have

$$|N(x) - N_{u(x,t)}(x)| \le C|\nabla(\phi - \psi_m)(x)|.$$

Using (6.8) and (6.5) we get

$$|N(x) - N_{u(x,t)}(x)| \le C(\phi(x) - t)^{(3p+1)/(p-1)} \le Cc\eta < \eta/3$$

if $c < 1/(3C)$; thus (6.6) follows. The proof of (6.7) follows similarly, making use of (6.9).

LEMMA 6.2. *There exist positive constants c_0, c_1, C_0 such that for any $\eta \in (0,1)$ such that*

$$\varepsilon^2 < c_0\eta, \tag{6.10}$$

the following is true: if

$$2M\varepsilon \le \phi(x) - t < c\eta^{(p-1)/(3p+1)}, \tag{6.11}$$

$$\tau \in S_{2\eta/3}(x), \qquad x \in B_R, \tag{6.12}$$

then

$$\frac{\partial u_\varepsilon(x,t)}{\partial \tau} \ge \frac{c_1\eta}{(\phi(x) - t)^{pq-1}}, \tag{6.13}$$

$$\frac{\partial}{\partial t}\frac{\partial u_\varepsilon(x,t)}{\partial \tau} \ge \frac{c_1\eta}{(\phi(x) - t)^{pq}}, \tag{6.14}$$

$$\left|\nabla\frac{\partial u_\varepsilon(x,t)}{\partial \tau}\right| \le \frac{C_0}{(\phi(x) - t)^{pq}} \tag{6.15}$$

provided ε is sufficiently small.

PROOF. The estimate (6.15) follows from Lemma 5.2. Next, from Lemma 6.1 we have, if $\tau \in S_{2\eta/3}(x)$,

$$\frac{\partial u(x,t)}{\partial \tau} \ge \frac{\eta}{3}\frac{\partial u}{\partial N_u}, \qquad \frac{\partial u}{\partial t}\frac{\partial u(x,t)}{\partial \tau} \ge \frac{\eta}{3}\frac{\partial u_t}{\partial M_{u_t}}.$$

We also clearly have

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial u}{\partial N_u}, \qquad \frac{\partial^2 u}{\partial t^2} = \beta \frac{\partial u_t}{\partial M_{u_t}}$$

where

$$\alpha = \alpha(x,t) = [1 + |\nabla \phi_m(x)|^2]^{-1/2}, \quad m = u(x,t),$$
$$\beta = \beta(x,t) = [1 + |\nabla \psi_m(x)|^2]^{-1/2}, \quad m = u_t(x,t).$$

Recalling the estimates in (2.11), we conclude that

$$\frac{\partial u}{\partial \tau} \geq \frac{c\eta}{(\phi - t)^{pq-1}}, \quad \frac{\partial}{\partial t}\frac{\partial u}{\partial \tau} \geq \frac{c\eta}{(\phi - t)^{pq}} \qquad (c > 0).$$

If we now make use of Lemma 5.3, we obtain from the last two inequalities the inequalities (6.13), (6.14) provided ε^2/η is bounded by a sufficiently small constant, i.e., provided (6.10) holds with c_0 small enough (independently of η, ε). This completes the proof of the lemma.

We now proceed to establish (1.17). Fix a point y in B_R (R small) and let $t_1 = \phi(y) - 3M\varepsilon$. Denote by K the cone with base

$$B \equiv \{(x, t_1); |x - y| < 5M\varepsilon^2\}$$

and vertex $(y, \phi(y) + 2M\varepsilon)$.

If $\tau \in S_\eta(y)$ then (since ϕ is smooth) $\tau \in S_{2\eta/3}(x)$ for any $x \in B_{5M\varepsilon^2}(y)$ provided

$$\varepsilon^2 \leq \eta/C \tag{6.16}$$

and C is a sufficiently large positive constant. It follows that (6.13)–(6.15) hold on B. We can therefore apply a comparison argument to $U_\varepsilon(x,t) = \partial u_\varepsilon(y + \varepsilon x, t)/\partial \tau$ (as in [**2**]; cf. also the proof of Lemma 5.4) and deduce that

$$\partial u_\varepsilon/\partial \tau > 0 \quad \text{in } K \cap \{t < \phi\} \tag{6.17}$$

provided (6.16) holds with C large enough. Since K contains $(y, \phi_\varepsilon(y))$ in its interior, we see from (6.17) that u_ε is increasing along any direction $\tau (\tau \in S_\eta(y))$, in some neighborhood of $(y, \phi_\varepsilon(y))$. This means that the direction of $\nabla \phi_\varepsilon(y)$ and $\nabla \phi(y)$ differ by at most η. Thus

$$|\nabla(\phi_\varepsilon - \phi)| \leq \eta \quad \text{at } y.$$

The constant η was subject only to the constraints (6.10), (6.11) for any $x \in B_{5M\varepsilon^2}(y)$ with $\phi(y) - t = 3M\varepsilon$, and (6.16). Thus we can choose $\eta = C\varepsilon^2$, where C is a sufficiently large positive constant, and then (1.17) follows.

REMARK 6.1. Using Lemma 5.4 we can extend the estimates of Lemma 5.3 to $t_1 < t < \phi(x) - M\varepsilon^2$, and of Lemma 6.2 under the condition $2M\varepsilon^2 < \phi(x) - t < c\eta^{(p-1)/(3p+1)}$ instead of (6.11). If we now follow the proofs of (1.16), (1.17) with these improved lemmas (using, for instance, in the proof of (1.17), cones with base B on $t = t_1 \equiv \phi(y) - 3M\varepsilon^2$ given by $B = \{(x, t_1); |x - y| < 5M\varepsilon^3\}$), we do not get any improvements of the estimates (1.16), (1.17).

7. The case $\phi(0) = \infty$. In this section we consider the case where $\phi(0) = \infty$ but $\phi(x) < \infty$ if $x \in B_R \backslash \{0\}$ for some $R > 0$. By Lemma 1.3 for any $x_0 \in B_R \backslash \{0\}$ there exists $\delta_0 = \delta_0(|x_0|)$ and $\varepsilon_0 = \varepsilon_0(\delta_0, |x_0|)$ positive such that there is a unique solution $u_\varepsilon(x,t)$ with blow-up surface $t = \phi_\varepsilon(x)$ if $x \in B_\delta(x, 0)$ and $0 < \varepsilon < \varepsilon_0$;

however ε_0 may go to zero if $x_0 \to 0$. On the other hand, for $N = 1$, ϕ_ε is finite for all $x \in B_R$ or even for all $x \in \mathbf{R}^1$ with $|\phi'_\varepsilon(x)| < 1/\varepsilon$ (by [**1**]).

For simplicity we take

$$f(u) = (u^+)^p, \qquad p > 1, \tag{7.1}$$

and consider first the case where

$$g(x) > 0 \quad \text{in } B_R\backslash\{0\}, \qquad g(x) = \sum_{i=1}^N \alpha_i x_i^2 + O(|x|^3) \quad \text{where } \alpha_i > 0\ \forall i, \tag{7.2}$$

$$h(x) > 0 \quad \text{in } B_R\backslash\{0\}, \qquad h(x) = \sum_{i=1}^N \beta_i x_i^2 + O(|x|^3) \quad \text{where } \beta_i > 0\ \forall i. \tag{7.3}$$

We further assume that if $N = 2$ or $N = 3$ then the solution $u_\varepsilon(x,t)$ with finite-valued blow-up surface $t = \phi_\varepsilon(x)$ exist for all $x \in B_R$.

THEOREM 7.1. *Under the foregoing assumptions, there exist positive constants* c_0, c_1 *such that*

$$c_0\varepsilon^{-\sigma} \le \phi_\varepsilon(x) \le c_1\varepsilon^{-\sigma} \quad \textit{where } \sigma = 2(p-1)/3p - 1. \tag{7.4}$$

PROOF. We shall construct a subsolution $v_\varepsilon(x,t)$ with blow-up surface $t = \psi_\varepsilon(x)$ such that

$$\psi_\varepsilon(0) = c_1\varepsilon^{-\sigma}. \tag{7.5}$$

Similarly we shall construct a supersolution $w_\varepsilon(t)$ with blow-up time T_ε,

$$T_\varepsilon \ge c_0\varepsilon^{-\sigma}. \tag{7.6}$$

Since $w_\varepsilon \ge u_\varepsilon > v_\varepsilon$,

$$\psi_\varepsilon(0) \ge T_\varepsilon \ge c_0\varepsilon^{-\sigma}, \tag{7.7}$$

and the proof of (6.4) will be completed.

Observe that the domain of dependence of a point $(0,T)$ with $T \sim c\varepsilon^{-\sigma}$ is a cone whose base on $\{t = 0\}$ is B_{R_0} where $R_0 \sim c\varepsilon^{1-\sigma}$. Therefore, in constructing v_ε and w_ε we need to define their Cauchy data only in a ball $\{|x| < C\varepsilon^{1-\sigma}\}$ with an appropriate positive constant C.

We take

$$v_\varepsilon(x,t) = (1+t)\delta|x|^2 + k(t) \tag{7.8}$$

where δ is a small positive constant and k is the solution of

$$k'' = k^p + 2\delta N\varepsilon^2 t \quad \text{if } t > 0, \qquad k(0) = k'(0) = 0. \tag{7.9}$$

It is easy to check that $\Box_\varepsilon v_\varepsilon \le v_\varepsilon^p$. In order to compare the Cauchy data, we work with $U_\varepsilon(x,t) = u_\varepsilon(\varepsilon x,t)$ and $V_\varepsilon(x,t) = v_\varepsilon(\varepsilon x,t)$. Denote by G_ε the function defined by (3.3) with $g = g(\varepsilon x)$, $h = h(\varepsilon x)$, and denote by $\tilde{G}$ the function defined by (3.3) with $g = h = \delta\varepsilon^2|x|^2$. Using the integral representations (3.1) for U_ε and V_ε, and noting that $G \ge \tilde{G}$ if δ is small enough, we deduce by comparison (cf. [**2**]) that $U_\varepsilon \ge V_\varepsilon$, i.e., $u_\varepsilon \ge v_\varepsilon$.

The function $K(t) = \lambda^{2/(1-p)} k(\lambda t)$ where $\lambda = \varepsilon^{-\sigma}$ satisfies

$$K'' = K^p + 2\delta N t, \qquad K(0) = K'(0) = 0$$

and it blows up in some finite time c_1. Therefore $k(t)$ blows up in time $c_1\varepsilon^{-\sigma}$, and (7.5) follows.

We next construct a supersolution $w_\varepsilon(t)$,

$$\begin{aligned} &w''_\varepsilon = w_2^p \quad \text{for } t > 0, \\ &w_\varepsilon(0) = w'_\varepsilon(0) = A\varepsilon^{2(1-\sigma)}, \qquad A > 0. \end{aligned} \tag{7.10}$$

If A is sufficiently large then, using the representation (3.1) for $U_\varepsilon(x,t)$ and $w_\varepsilon(t)$ we deduce that $w_\varepsilon \geq u_\varepsilon$ in $K^\varepsilon(0,T)$ for $T = c_1\varepsilon^{-\sigma}$. Thus it remains to prove that w_ε blows up in time $T_\varepsilon \geq c_0\varepsilon^{-\sigma}$ for some $0 < c_0 < c_1$.

Now, from (7.10) we get, by integration,

$$\frac{1}{2}w_{\varepsilon,t}^2 = \frac{w_\varepsilon^{p+1}}{p+1} + \frac{A^2}{2}\varepsilon^{4(1-\sigma)}(1 + o(1))$$

where $o(1) \to 0$ if $\varepsilon \to 0$; hence

$$T_\varepsilon = \int_{A\varepsilon^{2(1-\sigma)}}^{\infty} \frac{dw}{[2w^{p+1}/(p+1) + A^2\varepsilon^{4(1-\sigma)}]^{1/2}} \geq c_0\varepsilon^{-\sigma}.$$

We shall next consider the case where g and h vanish to higher order at $x = 0$. We take $N = 1$ and assume that

$$g(x) \geq 0, \ g''(x) \geq 0 \quad \text{for } |x| \text{ near } 0, \tag{7.11}$$

and

$$\begin{aligned} &h(x) = \beta|x|^n + O(|x|^{n+1}), \\ &h'(x) = n\beta|x|^{n-2}x + O(|x|^n), \qquad \beta > 0, \end{aligned} \tag{7.12}$$

where n is a positive number ≥ 2.

Set

$$\sigma = \{1 + (p+1)/n(p-1)\}^{-1}. \tag{7.13}$$

THEOREM 7.2. *Let $N = 1$ and let* (7.1), (7.11), (7.12) *hold. Then there exist positive constants c_0, c_1 such that* (7.4) *holds with σ given by* (7.13).

Notice that for $n = 2$ this is an improvement of Theorem 7.1 for $N = 1$, since the condition (7.11) is weaker than (7.2).

PROOF. We represent u_ε in the form

$$\begin{aligned} u_\varepsilon(x,t) = {} & \frac{1}{2}[g(x+\varepsilon t) + g(x-\varepsilon t)] + \frac{1}{2\varepsilon}\int_{x-\varepsilon t}^{x+\varepsilon t} h(y)\,dy \\ & + \frac{1}{2\varepsilon}\int_0^t \int_{x-\varepsilon(t-\tau)}^{x+\varepsilon(t-\tau)} (u^+)^p(y,\tau)\,dy\,d\tau. \end{aligned} \tag{7.14}$$

By differentiation,

$$\begin{aligned} u_{\varepsilon,t}(x,t) = {} & \frac{\varepsilon}{2}[g'(x+\varepsilon t) - g'(x-\varepsilon t)] + \frac{1}{2}[h(x+\varepsilon t) + h(x-\varepsilon t)] \\ & + \frac{1}{2}\int_0^t [(u^+)^p(x+\varepsilon(t-\tau),\tau) + (u^+)^p(x-\varepsilon(t-\tau),\tau)]\,d\tau. \end{aligned} \tag{7.15}$$

Using (7.14) and the assumption $g(x) \geq 0$ for x near 0, we get

$$u_\varepsilon(x, \varepsilon^{-\sigma}) \geq \frac{1}{2\varepsilon} \int_{x-\varepsilon^{1-\sigma}}^{x+\varepsilon^{1-\sigma}} h(y)\, dy \geq \beta \varepsilon^{-\sigma} \varepsilon^{n(1-\sigma)} \quad \text{if } |x| < R \tag{7.16}$$

provided R is small; α is some positive constant.

Next, since $g'' \geq 0$, we obtain from (7.15),

$$u_{\varepsilon,t}(x, \varepsilon^{-\sigma}) \geq \beta \varepsilon^{n(1-\sigma)}/4 \quad \text{if } |x| \leq R. \tag{7.17}$$

We now compare $u_\varepsilon(x, \varepsilon^{-\sigma} + t)$ with the solution $v_\varepsilon(t)$ of

$$v_\varepsilon'' = v_\varepsilon^p, \qquad t > 0,$$
$$v_\varepsilon(0) = v_\varepsilon'(0) = \gamma_0 \varepsilon^{n(1-\sigma)}, \qquad \gamma_0 = \beta/4.$$

This solution blows up at time $T(\gamma_0 \varepsilon^{n(1-\sigma)}, \ \gamma_0 \varepsilon^{n(1-\sigma)})$ where $T(\gamma, \delta)$ is defined by (2.3). It is easily computed that the blow-up time is bounded from above by

$$C(\varepsilon^{-n(1-\sigma)})^{(p-1)/(p+1)} \qquad (C \text{ positive constant}),$$

which is equal to $C\varepsilon^{-\sigma}$ (with another positive constant C), by the definition of σ in (7.13). Hence

$$\phi_\varepsilon(0) < \varepsilon^{-\sigma} + C_* \varepsilon^{-\sigma} = c_1 \varepsilon^{-\sigma} \qquad (c_1 = 1 + C). \tag{7.18}$$

In order to estimate $\phi_\varepsilon(0)$ from below it is sufficient (in view of (7.18)) to use the initial data only in the interval $\{|x| < c_1 \varepsilon^{1-\sigma}\}$. Thus we can use the same function w_ε as in (7.10) for a supersolution. Proceeding as in the argument following (7.10), we then derive the lower estimate in (7.4) with σ as in (7.13).

REMARK. The above proof extends to $N = 2, 3$ and say

$$g(x) \sim \alpha |x|^n, \quad h(x) \sim \beta |x|^n \qquad (\alpha > 0, \ \beta > 0) \tag{7.19}$$

provided we already know that

$$u_\varepsilon(x,t) \geq 0, \qquad u_{\varepsilon,t}(x,t) \geq 0 \tag{7.20}$$

for $|x| \leq C\varepsilon^{1-\sigma}$, $0 \leq t \leq \varepsilon^{-\sigma}$; we use here the representation (3.1) for both u_ε and $u_{\varepsilon,t}$. However the usual method for proving (7.20) (by using the representation (3.1), as in [**2**]) does not extend to the present situation where (7.19) holds, even for t small.

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